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INDUCTION ON CLOSED, BOUNDED-BELOW SUBSETS OF $\mathbb{R}$

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ABSTRACT: Mathematical Induction, usually used to prove theorems of the form $\forall n \in \mathbb{N}(Q(n))$ for a proposition Q, can also be employed to prove theorems of the form $\forall x \in K(Q(x))$ for certain Q and for $K \subset \mathbb{R}$ such that K is closed and bounded below. The induction theorem is presented and applied to give direct proofs of several comparison theorems for integral and differential inequalities.

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1. INTRODUCTION

Applications of mathematical induction permeate many classical mathematical disciplines such as set theory (Suppes [5]), analysis (Royden [4]), algebra (van der Waerden [6]), and logic (Shoenfield [3]). Induction is generally used to establish conjectures of the following form: The proposition $Q(n)$ is true for all n in the set of natural numbers $\mathbb{N}$. Standard notation tacitly utilized here for this statement is $\forall n \in \mathbb{N}\ (Q(n))$, where "$Q(n)$" means "$Q(n)$ is true". Two common forms of induction are "weak" and "strong" (or "complete") induction (Mendelson [2]):

Weak Induction: If $Q(1)$ and for all $k \in \mathbb{N}$, $Q(k+1)$ whenever $Q(k)$, then

$$\forall n \in \mathbb{N}\ \ (Q(n)).$$

Weak Induction implies Strong Induction:

Strong Induction: If for all $n \in \mathbb{N}$, $Q(n)$ holds whenever $Q(m)$ for all $m \in \mathbb{N}$ such that $m < n$, then $\forall n \in \mathbb{N}(Q(n))$.

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Induction works for the natural numbers because of special ordering properties on $\mathbb{N}$ which ensure that each $n \in \mathbb{N}$ has a next largest (successor) element $n+1$. The Property of Transfinite Induction (see e.g. Mendelson [2]) generalizes strong induction to well-ordered sets (linearly ordered sets for which every nonempty subset has a least element):

Property of Transfinite Induction: Let $<$ be a well-ordering for the set W. Suppose that for all $w \in W$, we have $Q(w)$ whenever $Q(y)$ for all $y \in W$ such that $y < w$. Then $\forall w \in W(Q(w))$.

Proof: Define $\tilde{Q}(w)$ as the statement "not $Q(w)$". Suppose that for some $z \in W$, $\tilde{Q}(z)$. Then the truth set $\{z \in W | \tilde{Q}(z)\}$ is a nonempty subset of W, and hence contains a least element w. Thus, for all $y < w$, $Q(y)$, and so $Q(w)$, which is a contradiction. □

In general, induction cannot be used to establish results of the form $\forall x \in \mathbb{R}(Q(x))$ because the set of real numbers $\mathbb{R}$ is not well-ordered under the usual order "less than". For example, even though the proposition $x \leq 3$ holds whenever $y \leq 3$ for all $y < x$, it is not the case that $x \leq 3$ for all $x \in \mathbb{R}$.

There are, however, ways to use induction on certain closed subsets of $\mathbb{R}$ for certain predicates Q.

Theorem 1: *Let $a \in \mathbb{R}$. Suppose the truth set $A = \{t \in \mathbb{R} | Q(t))$ is an open set in $\mathbb{R}$ and, for all $t \in [a, \infty)$, $Q(t)$ whenever $Q(x)$ for all $x \in [a, t)$. Then*

$$\forall t \in [a, \infty)(Q(t)).$$

Proof: Suppose that $\tilde{Q}(t)$ for some $t \in [a, \infty)$. Then the set $(\mathbb{R} - A) \cap [a, \infty)$ is nonempty, closed, and bounded below, and thus contains its infimum T. Hence, for all $x \in [a, T]$, we have $Q(x)$, and so $Q(T)$, which is a contradiction. □

Note that the second hypothesis of Theorem 1 vacuously implies $Q(a)$. In applications, one must take care to verify this case.

Theorem 1 can easily be extended to any subset of $\mathbb{R}$ which is closed and bounded below:

Theorem 2: *Suppose $K \subset \mathbb{R}$ is closed and bounded below in $\mathbb{R}$; $A = \{t \in \mathbb{R} | Q(t)\}$ is open in $\mathbb{R}$; and for all $t \in K$, $Q(t)$, whenever $Q(x)$ for all $x \in K$ such that $x < t$. Then*

$$\forall t \in K(Q(t)).$$

2. APPLICATIONS

To illustrate applicable directions in which induction on closed, bounded below subsets of $I\!R$ may be utilized, we focus on some classical problems of differential and integral inequalities. Theorem 1 can be used to give direct proofs of many general comparison theorems for integral and delay-differential inequalities. These comparison theorems are commonly demonstrated by contradiction, with each proof reiterating a particular incarnation of the proof of Theorem 1. Besides abstracting a method of direct proof, Theorem 1 also provides a certain intuition for the truth of such comparison results. The next example concerns one of the basic comparison theorems for integral inequalities.

Example 1 (Comparison Theorem for Integral Inequalities):

Theorem (Lakshmikantham and Leela [1]): *Let* $J = [a, \infty)$, $K \in C[J \times J \times I\!R, I\!R]$, $x, y, f \in C[J, I\!R]$, *and* $K(s,t,x)$ *be strictly increasing in* x *for each fixed* (s,t). *If*

i) $x(t) \leq f(t) + \int_a^t K(s,t,x(s))ds \quad \forall t \in [a, \infty)$,
ii) $y(t) \geq f(t) + \int_a^t K(s,t,y(s))ds \quad \forall t \in [a, \infty)$,
iii) $x(a) < y(a)$,

then

$$x(t) < y(t) \quad \forall t \geq a.$$

Proof: Continuously extend x and y to $I\!R$ by defining $x(t) = x(a)$ and $y(t) = y(a)$ for $t < a$. Then the truth set $\{t \in I\!R | x(t) < y(t)\}$ is open in $I\!R$.

Let $T \in [a, \infty)$. If $T = a$, then $x(T) < y(T)$. Otherwise, assume $x(t) < y(t)$ for all $t \in [a, T)$. Then

$$x(T) \leq f(T) + \int_a^T K(s,T,x(s))ds$$

$$< f(T) + \int_a^T K(s,T,y(s))ds \leq y(T).$$

□

Example 2 (Comparison Theorem for Delay-Differential Equations):

Theorem: *Let* $J = [t_0, \infty)$, $f \in C[J \times I\!R, I\!R]$, $x, y \in C^1[I\!R, I\!R]$, y *be nondecreasing, and* $f(t,x)$ *be nondecreasing in* x *for each fixed* t. *If for fixed* $\alpha \in (0, \infty)$

i) $x'(t) \leq f(t, x(t-\alpha)) \quad \forall t \geq t_0$,

ii) $y'(t) \geq f(t, y(t)) \ \ \forall t \geq t_0$,
iii) $x(t) < y(t) \ \ \forall t \in [t_0 - \alpha, t_0]$

then

$$x(t), y(t) \ \ \forall t \geq t_0.$$

Proof: The truth set $\{t \in I\!R | x(t) < y(t)\}$ is open in $I\!R$.

Let $T \in [t_0, \infty)$. If $T = t_0$, then $x(T) < y(T)$. Otherwise, assume $x(t) < y(t)$ for all $t \in [t_0, T)$. If $\tau \in [t_0, T]$, then

$$x'(\tau) \leq f(\tau, x(\tau - \alpha)) \leq f(\tau, y(\tau - \alpha)) \leq f(\tau, y(\tau)) \leq y'(\tau),$$

and hence $x'(t) \leq y'(t)$ for all $t \in [t_0, T]$. Thus, $\int_{t_0}^{T} x'(s)ds \leq \int_{t_0}^{T} y'(s)ds$, and so $x(T) - x(t_0) \leq y(T) - y(t_0)$. By iii), $x(T) < y(T)$. □

The final example establishes the asymptotic behavior of solutions to an integral inequality, given that the kernel is dominated by a function of a particular form.

Example 3: Let g be positive and continuous on $[0, \infty) \times [, \infty)$, and $k > 0$. Positive continuous solutions of the integral inequality

$$x(t) \leq \int_0^t x(t - a) g(a, t) da + \frac{1}{2} k^{t+1}, \ \ t \in [0, \infty)$$

are of exponential order as $t \to \infty$ provided $g(a, t) \leq \frac{k^a}{2t}$ everywhere in $[0, \infty) \times (0, \infty)$. In particular, solutions decay exponentially if $0 < k < 1$.

We will show that $x(t) < k^{t+1}$ for all $t \in [0, \infty)$.

Extend x to $I\!R$ by defining $x(t) = x(0)$ for all $t < 0$. Then the truth set $\{t \in I\!R | x(t) < k^{t+1}\}$ is open in $I\!R$ since x is continuous. Let $T \in [0, \infty)$. If $T = 0$, then $x(T) = x(0 \leq \frac{1}{2}k < k = k^{T+1}$. Otherwise, assume that $x(t) < k^{t+1}$ whenever $t \in [0, T)$. Then

$$\begin{aligned} x(T) &\leq \int_0^T x(T - a) g(a, T) da + \frac{1}{2} k^{T+1} \\ &< k^{T+1} \int_0^T \frac{1}{2T} da + \frac{1}{2} k^{T+1} \\ &= k^{T+1}. \end{aligned}$$

□

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